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Effective thermoelastic properties of graded doubly periodic particulate matrix composites in varying external stress fields

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Abstract

We consider a linear elastic composite medium, which consists of a homogeneous matrix containing aligned ellipsoidal uncoated or coated inclusions arranged in a doubly periodic array and subjected to inhomogeneous boundary conditions. The hypothesis of effective field homogeneity near the inclusions is used. The general integral equation obtained reduces the analysis of infinite number of inclusion problems to the analysis of a finite number of inclusions in some representative volume element (RVE). The integral equation is solved by a modified version of the Neumann series; the fast convergence of this method is demonstrated for concrete examples. The nonlocal macroscopic constitutive equation relating the cell averages of stress and strain is derived in explicit iterative form of an integral equation. A doubly periodic inclusion field in a finite ply subjected to a stress gradient along the functionally graded direction is considered. The stresses averaged over the cell are explicitly represented as functions of the boundary conditions. Finally, the employed of proposed explicit relations for numerical simulations of tensors describing the local and nonlocal effective elastic properties of finite inclusion plies containing a simple cubic lattice of rigid inclusions and voids are considered. The local and nonlocal parts of average strains are estimated for inclusion plies of different thickness. The boundary layers and scale effects for effective local and nonlocal effective properties as well as for average stresses will be revealed.

1. Introduction

The problem to be discussed is the determination of the overall response of linear elastic materials composed of a homogeneous matrix containing identical coated inclusions arranged in some

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doubly periodic array. The doubly periodic structure of composite materials is very attractive both because it provides the estimation of interaction effects for an infinite number of inclusions and because the breakdown of the periodicity in one direction leading to such structures can be considered as a model of so-called Functionally Graded Materials (FGMs). FGMs are the materials which feature gradual compositional or microstructural transitions, being designed to deliver in an optimal way certain functional performance requirements that vary with location within a part. The FGMs approach is, by its own intrinsic nature, far more multidisciplinary than almost any other undertaking in material research. In particular cases of the above problems it is possible to use different generalizations of the known methods for triply periodic structures (see e.g. Kuznetsov, 1991; Buryachenko and Parton, 1992; Rodin, 1993; Kushch, 1997; Nemat-Nasser and Hori, 1993).

At the present time the homogenization theory for composites with regular structure is developed in detail (references can be found e.g. in Sanchez-Palencia, 1980; Bakhvalov and Panasenko, 1989; Kalamkarov and Kolpakov, 1997). Most of these works are based on the use of multivariable asymptotic techniques. If the unit cell is much smaller than the size of the structure the variation of stress and strain states from cell to cell will be small. For numerical solutions of homogenization theory the number of the inclusions in the unit cell can be increased significantly by increasing the size of the unit cell as compared with inclusion size; so Nakamura and Suresh (1993) considered 30 and 60 fibers in one unit cell. In particular, the modeling of FGM, Weissenbek et al. (1997) considered a unit cell as a column containing an inclusion set in one side of the column only. Pindera et al. (1995) used an analogous scheme with an approximate representation of the local stress states by second-order polynomials in the neighborhood of each inclusion. In practice, however, components may be subjected to nonuniform stress states, such as in a circumferential reinforced ring subjected to radial turbine blade loads and centrifugal inertial loads (see e.g. Du et al., 1995).

At the same time for random structure composites it is well known, that the eventual abandonment of the so-called hypothesis of statistically homogeneous fields leads to a nonlocal coupling between statistical averages of the strain $\langle \boldsymbol{\varepsilon} \rangle(\mathbf{x})$ and stress $\langle \boldsymbol{\sigma} \rangle(\mathbf{x})$ tensors when the statistical average stress is given by an integral of the field quantity weighted by some tensorial function, i.e. the nonlocal effective compliance $\mathscr{M}^*$:

$$\langle \boldsymbol{\varepsilon} \rangle(\mathbf{x}) = \int \mathscr{M}^*(\mathbf{x}, \mathbf{y}) \langle \boldsymbol{\sigma} \rangle(\mathbf{y})\, \mathrm{d}\mathbf{y}. \tag{1.1}$$

In the consideration of dispersed media with both regular and random structures this approach makes intuitive sense since the stress at any point will depend on the arrangement of the surrounding inclusions. Therefore, the value of the statistical average field for random structures (or the values of the average field over the cell for periodic structures) will locally depend on its value at the other points in its vicinity. This is especially true if the inclusion number density varies over distances that are comparable to the particle size. The method of Fourier transforms has been much investigated in nonlocal micromechanics of random structure composites and was used with some modifications by Beran and McCoy (1970), Buryachenko and Lipanov (1992), Drugan and Willis (1996), Khoroshun (1996), Buryachenko (1998). The same approach can be employed for the nonlocal analysis of triply periodic structures (see e.g. Buryachenko, 1998). Nevertheless, using the Fourier transform method in the case of doubly periodic structures leads to the principal difficulties because the general integral equation being analyzed in the current paper is not a convolution equation, and therefore the standard Fourier transform properties for the transform of both convolution integrals and derivatives cannot be used. For elimination of these difficulties we will propose modified versions of the iteration method.

The outline of the paper is as follows. In Section 2 we present the basic equation and geometrical description of the composite structure, and two kinds of averaging operators. In Section 3 we derive a general integral equation of elasticity of general doubly periodic structures subjected to inhomogeneous boundary conditions; the equations obtained reduce the analyses of problems infinite numbers of inclusions to the analysis of a finite number of inclusions. This equation is simplified for particular cases of triply periodic structures under homogeneous boundary conditions and for inclusion fields bounded in one direction. In order to simplify the general integral system one uses the effective field hypothesis. In Section 4 the Fredholm integral equation of the second kind obtained in the paper is solved by a modified version of the iteration method of the Neumann series; the fast convergence of this method is demonstrated for concrete examples. Once the average effective field is obtained, the effective local and nonlocal constitutive relations of the composite material are calculated via the homogenized relation in Section 5. A doubly periodic inclusion field in a finite ply subjected to a stress gradient along the functionally graded direction is considered in Section 6. The stresses averaged over the cell are explicitly represented as functions of the boundary conditions. Finally, in Section 7 we employ the proposed explicit relations for numerical simulations of tensors describing the local and nonlocal effective elastic properties of finite inclusion plies containing a simple cubic lattice of rigid inclusions and voids. The local and nonlocal parts of average strains are estimated for inclusion plies of different thickness. The boundary layers and scale effects for effective local and nonlocal effective properties as well as for average stresses will be revealed.

2. Preliminaries

2.1. Basic equations

The paper discusses a certain representative mesodomain w with a characteristic function W containing a doubly periodic set $X=(v_i)$ of inclusions v_i with characteristic functions V_i $(i=1,2,\ldots)$. At first no restrictions are imposed on the elastic symmetry of the phases or on the geometry of the inclusions. It is assumed that the inclusions can be grouped into the component $v^{(1)}$ with identical mechanical and geometrical properties. The local strain tensor $\boldsymbol{\varepsilon}$ is related to the displacements $\mathbf{u}$ via the linearized strain–displacement equation

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}[\boldsymbol{\nabla}\otimes\mathbf{u}+(\boldsymbol{\nabla}\otimes\mathbf{u})^{\mathrm{T}}]. \tag{2.1}$$

Here $\otimes$ denotes tensor product, and $(\cdot)^{\mathrm{T}}$ denotes matrix transposition. The stress tensor $\boldsymbol{\sigma}$, satisfies the equilibrium equation (no body forces acting):

$$\boldsymbol{\nabla}\boldsymbol{\sigma} = 0. \tag{2.2}$$

Stresses and strains are related to each other via the constitutive equations

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{L}(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{x})+\boldsymbol{\alpha}(\mathbf{x}) \quad \text{or } \boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{M}(\mathbf{x})\boldsymbol{\sigma}(\mathbf{x})+\boldsymbol{\beta}(\mathbf{x}). \tag{2.3}$$

$\mathbf{L}(\mathbf{x})$ and $\mathbf{M}(\mathbf{x}) \equiv \mathbf{L}(\mathbf{x})^{-1}$ are the known phase stiffness and compliance fourth-order tensors, and the common notations for scalar products have been employed: $\mathbf{L}\boldsymbol{\varepsilon} = L_{ijkl}\varepsilon_{kl}$, $\boldsymbol{\beta}(\mathbf{x})$ and $\boldsymbol{\alpha}(\mathbf{x}) \equiv -\mathbf{L}(\mathbf{x})\boldsymbol{\beta}(\mathbf{x})$ are second-order tensors of local eigenstrains and eigenstresses (frequently called transformation fields), respectively, which may arise by thermal expansion, phase transformation, twinning and other changes of shape or volume of the material. All tensors $\mathbf{f}$ ($\mathbf{f} = \mathbf{L}, \mathbf{M}, \boldsymbol{\alpha}, \boldsymbol{\beta}$) of material properties are decomposed as $\mathbf{f} \equiv \mathbf{f}^{(0)} + \mathbf{f}_1(\mathbf{x})$. $\mathbf{f}$ is assumed to be constant in the matrix $v^{(0)} = w \backslash v$ and is an inhomogeneous function inside the inclusions:

$$\mathbf{f}(\mathbf{x}) = \begin{cases} \mathbf{f}^{(0)} & \text{for } \mathbf{x} \in v^{(0)}, \\ \mathbf{f}^{(0)} + \mathbf{f}_1^{(k)}(\mathbf{x}) & \text{for } \mathbf{x} \in v^{(k)}. \end{cases} \tag{2.4}$$

Here and in the following the upper index (k) numbers the components and the lower index i numbers the individual inclusions; $v \equiv \bigcup v^{(k)} \equiv \bigcup v_i$, $(k = 1, 2, \ldots, N; i = 1, 2, \ldots)$.

We assume that the phases are perfectly bonded, so that the displacements and the traction components are continuous across the interphase boundaries. We take nonuniform traction boundary conditions for the mesodomain w

$$\boldsymbol{\sigma}^0(\mathbf{x})\mathbf{n}(\mathbf{x}) = \mathbf{t}(\mathbf{x}), \quad \mathbf{x} \in \partial w, \tag{2.5}$$

where $\mathbf{t}(\mathbf{x})$ is the traction vector at the external boundary ∂w, $\mathbf{n}$ is its unit outward normal, and $\boldsymbol{\sigma}^0(\mathbf{x})$ is a given symmetric tensor, representing the macroscopic stress state in the mesodomain w if the boundary conditions (2.5) is a homogeneous one: $\boldsymbol{\sigma}^0(\mathbf{x}) \equiv$ const., $\mathbf{x} \in \partial w$. It is assumed that the points under consideration are not close to the boundary ∂w.

2.2. Geometrical description of the composite structure

It is assumed that the representative mesodomain w contains a statistically large number of inclusions $v_i \subset v^{(1)}$ $(i = 1, 2, \ldots)$. We now consider a doubly periodic set X of ellipsoidal inclusions with identical shape, orientation and mechanical properties. Suppose $\mathbf{e}_i$ $(i = 1, 2, 3)$ are linearly-independent vectors, so that we can represent any node $\mathbf{m} \in \Lambda$

$$\mathbf{x}_{\mathbf{m}} = m_1\mathbf{e}_1 + m_2\mathbf{e}_2 + f(m_3)\mathbf{e}_3, \tag{2.6}$$

where $\mathbf{m} = (m_1, m_2, m_3)$ are integer-valued coordinates of the node $\mathbf{m}$ in the basis $\mathbf{e}_i$ which are equal in modulus to $|\mathbf{e}_i|$, and $f(m_3) - f(m_3 + 1) \neq$ const. In the plane $f(m_3) =$ const. the composite is reinforced by periodic arrays Λ_{m_3} of inclusions in the direction of the $\mathbf{e}_1$-axis and the $\mathbf{e}_2$-axis. The type of the lattice Λ_{m_3} is defined by the law governing the variation in the coefficients m_i $(i = 1, 2)$, and also by the magnitude and orientation of the vectors $\mathbf{e}_i$ $(i = 1, 2)$. In the functionally graded direction $\mathbf{e}_3$ the inclusion spacing between adjacent arrays may vary $(f(m_3) - f(m_3 + 1) \neq$ const.). For a doubly-periodic array of inclusions in a finite ply containing $2m^l + 1$ layers of inclusions we have $f(m_3) \equiv 0$ at $|m_3| > m^l$ (see Fig. 1); in more general cases of doubly periodic structures $f(m_3) \neq 0$ at $m_3 \to \pm\infty$. To make exposition more clear we will assume that the basis $\mathbf{e}_i$ is an orthogonal one and the axes $\mathbf{e}_i$ $(i = 1, 2, 3)$ are directed along axes of global Cartesian coordinate system (these assumptions are not obligatory).

The composite material is constructed using the building blocks or cells: $w = \bigcup \Omega_{\mathbf{m}}$, $v_{\mathbf{m}} \subset \Omega_{\mathbf{m}}$.

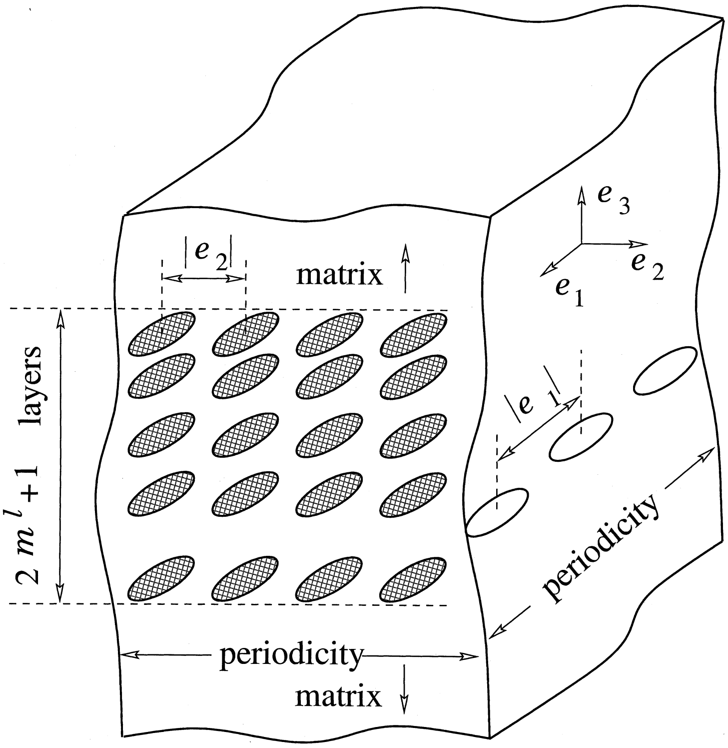

Fig. 1. Schematic representation of the doubly periodic inclusion ply.

Hereafter the notation $\mathbf{f}^{\Omega}(\mathbf{x})$ will be used for the average of the function $\mathbf{f}$ over the cell $\mathbf{x} \in \Omega_i$ with the center $\mathbf{x}_i^{\Omega} \in \Omega_i$:

$$\mathbf{f}^{\Omega}(\mathbf{x}) = \mathbf{f}^{\Omega}(\mathbf{x}_i^{\Omega}) \equiv n(\mathbf{x}) \int_{\Omega_i} \mathbf{f}(\mathbf{y})\,\mathrm{d}\mathbf{y}, \quad \mathbf{x} \in \Omega_i, \tag{2.7}$$

$n(\mathbf{x}) \equiv 1/\bar{\Omega}_i$ is the number density of inclusions in the cell Ω_i.

Let $\mathscr{V}_{\mathbf{x}}$ be a 'moving averaging' cell with the center $\mathbf{x}$ and characteristic size $a_{\mathscr{V}} = \sqrt[3]{\bar{\mathscr{V}}}$, and let for the sake of definiteness $\boldsymbol{\xi}$ be a random vector uniformly distributed on $\mathscr{V}_{\mathbf{x}}$ whose value at $\mathbf{z} \in \mathscr{V}_{\mathbf{x}}$ is $\varphi_{\xi}(\mathbf{z}) = 1/\bar{\mathscr{V}}_{\mathbf{x}}$ and $\varphi_{\xi}(\mathbf{z}) \equiv 0$ otherwise. Then we can define the average of the function $\mathbf{f}$ with respect to translations of the vector $\boldsymbol{\xi}$

$$\langle \mathbf{f} \rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y}) = \frac{1}{\bar{\mathscr{V}}_{\mathbf{x}}} \int_{\mathscr{V}_{\mathbf{x}}} \mathbf{f}(\mathbf{z}-\mathbf{y})\,\mathrm{d}\mathbf{z}, \quad \mathbf{x} \in \Omega_i. \tag{2.8}$$

Among other things 'moving averaging' cell $\mathscr{V}_{\mathbf{x}}$ can be obtained by translation of a cell Ω_i and can vary in size and shape during the motion from point to point. Clearly, contracting the cell $\mathscr{V}_{\mathbf{x}}$ to the point $\mathbf{x}$ occurs in passing to the limit $\langle \mathbf{f} \rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y}) \to \mathbf{f}(\mathbf{x}-\mathbf{y})$. To make the exposition more clear we will assume that $\mathscr{V}_{\mathbf{x}}$ results from Ω_i by translation of the vector $\mathbf{x}-\mathbf{x}_i^{\Omega}$; it can be seen, however, that this assumption is not obligatory.

By way of illustration let us consider the case of triply periodic structures under the uniform boundary conditions:

$$f(m_3)-f(m_3+1) \equiv \text{const.}, \quad \forall m_3, \tag{2.9}$$

$$\boldsymbol{\sigma}^0(\mathbf{x}) = \boldsymbol{\sigma}^0 \equiv \text{const.}, \tag{2.10}$$

and let $\mathbf{f}$ be governed by the boundary condition (2.10) (for example $\mathbf{f} \equiv \boldsymbol{\sigma}$). Clearly for homogeneous boundary conditions (2.10), $\boldsymbol{\sigma}^{\Omega}(\mathbf{x})$ by (2.7) is an invariant with respect to the cell number i and $\boldsymbol{\sigma}^{\Omega}(\mathbf{x}) = \langle\boldsymbol{\sigma}\rangle_{\mathrm{x}}(\mathbf{x}) = \text{const.}, \forall \mathbf{x} \in \Omega_i \subset w$ (if $\mathscr{V}_{\mathbf{x}}$ is a translation of Ω_i). In the general case of inhomogeneous boundary conditions $\boldsymbol{\sigma}^0(\mathbf{x}) \neq \text{const.}$ (2.5) (as well as in the case that the condition (2.9) breaks down) $\boldsymbol{\sigma}^{\Omega}(\mathbf{x})$ is a step function $\boldsymbol{\sigma}^{\Omega}(\mathbf{x}) \neq \boldsymbol{\sigma}^{\Omega}(\mathbf{y})$ at $\mathbf{x} \in \Omega_i$ and $\mathbf{y} \in \Omega_j$ $(i \neq j)$ as well as $\boldsymbol{\sigma}^{\Omega}(\mathbf{x}) \neq \langle\boldsymbol{\sigma}\rangle_{\mathrm{x}}(\mathbf{x})$ at $\mathbf{x} \in \Omega_i$.

3. General integral equations and effective field hypothesis

3.1. General integral equations

From eqns (2.1)–(2.4) a general integral equation for $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ can be derived. Substituting (2.3) and (2.1) into the equilibrium eqn (2.2), we obtain a differential equation for the strain $\boldsymbol{\varepsilon}$

$$\nabla(\mathbf{L}^{(0)}+\mathbf{L}_1(\mathbf{x}))[\boldsymbol{\varepsilon}(\mathbf{x})-(\boldsymbol{\beta}^{(0)}+\boldsymbol{\beta}_1(\mathbf{x}))] = 0. \tag{3.1}$$

Introducing the modified strains $\mathbf{e}(\mathbf{x}) \equiv \boldsymbol{\varepsilon}(\mathbf{x})-\boldsymbol{\beta}^{(0)}$, eqn (3.1) may be reduced to a symmetrized integral form

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \boldsymbol{\varepsilon}^0(\mathbf{x})-\nabla\int \mathbf{G}(\mathbf{x}-\mathbf{y})\{\nabla\mathbf{L}_1(\mathbf{y})\mathbf{e}(\mathbf{y})-\nabla[\mathbf{L}^0+\mathbf{L}_1(\mathbf{x})]\boldsymbol{\beta}_1(\mathbf{x})\}\,d\mathbf{y}, \tag{3.2}$$

where $\boldsymbol{\varepsilon}^0(\mathbf{x}) \equiv \mathbf{M}^{(0)}\boldsymbol{\sigma}^{(0)}(\mathbf{x})$ is the modified strain which would exist in the medium under the same boundary conditions if $\mathbf{L} \equiv \mathbf{L}^{(0)}$ and $\boldsymbol{\beta} \equiv \mathbf{0}$. $\mathbf{G}$ is the infinite-homogeneous-body Green's tensor of the Lamé's equation for a homogeneous medium with an elastic modulus tensor $\mathbf{L}^{(0)}$

$$\nabla\{\mathbf{L}^{(0)}\tfrac{1}{2}[\nabla\otimes\mathbf{G}(\mathbf{x})+(\nabla\otimes\mathbf{G}(\mathbf{x}))^{\mathrm{T}}]\} = -\boldsymbol{\delta}\delta(\mathbf{x}), \tag{3.3}$$

$\delta(\mathbf{x})$ is the Dirac delta function, $\boldsymbol{\delta}$ is the unit second-order tensor.

After integration of eqn (3.2) by parts, it is found that

$$\mathbf{e}(\mathbf{x}) = \mathbf{e}^0(\mathbf{x})+\int \mathbf{U}(\mathbf{x}-\mathbf{y})\{\mathbf{L}_1(\mathbf{y})\mathbf{e}(\mathbf{y})-[\mathbf{L}^{(0)}+\mathbf{L}_1(\mathbf{y})]\boldsymbol{\beta}_1(\mathbf{y})\}\,d\mathbf{y}$$
$$+\oint\nabla\mathbf{G}(\mathbf{x}-\mathbf{s})\{\mathbf{L}_1(\mathbf{s})\mathbf{e}(\mathbf{s})-[\mathbf{L}^{(0)}+\mathbf{L}_1(\mathbf{s})]\boldsymbol{\beta}_1(\mathbf{s})\}\mathbf{n}(\mathbf{s})\,d\mathbf{s}, \tag{3.4}$$

where the surface integration is taken over the boundary $\mathbf{s} \in \partial w$ of the mesodomain w, containing a statistically large number of inclusions; $\mathbf{n}$ is the unit outward normal. The integral operator kernel $\mathbf{U}$ is defined by the Green tensor $\mathbf{G}$ (3.3):

$$U_{ijkl}(\mathbf{x}) = [\nabla_j\nabla_l G_{ik}(\mathbf{x})]_{(ij)(kl)}, \tag{3.5}$$

where the notation indicates symmetrization on (ij) and (kl).

Equation (3.4) is centered, i.e. from both sides of eqn (3.4) their average over the moving averaging cell $\mathscr{V}_{\mathbf{x}}$ (2.8) are subtracted

$$\mathbf{e}(\mathbf{x}) = \langle\langle \mathbf{e} \rangle\rangle_{\mathbf{x}}(\mathbf{x}) + \langle \mathbf{e} \rangle_{\mathbf{x}}(\mathbf{x}) + \int \langle\langle \mathbf{U} \rangle\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y})\{\mathbf{L}_1(\mathbf{y})\mathbf{e}(\mathbf{y}) - [\mathbf{L}^{(0)} + \mathbf{L}_1(\mathbf{y})]\boldsymbol{\beta}_1(\mathbf{y})\}\,\mathrm{d}\mathbf{y} + \oint \langle\langle \nabla \mathbf{G} \rangle\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{s})\{\mathbf{L}_1(\mathbf{s})\mathbf{e}(\mathbf{s}) - [\mathbf{L}^{(0)} + \mathbf{L}_1(\mathbf{s})]\boldsymbol{\beta}_1(\mathbf{s})\}\mathbf{n}(\mathbf{s})\,\mathrm{d}\mathbf{s}, \quad (3.6)$$

where $\mathbf{x} \in \Omega_i$, and one introduces a new centering operation over the cell $\mathbf{x} \in \boldsymbol{\Omega}_i$

$$\langle\langle \mathbf{f} \rangle\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y}) \equiv \mathbf{f}(\mathbf{x}-\mathbf{y}) - \langle \mathbf{f} \rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y}). \quad (3.7)$$

For the analyses of integral convergence in eqn (3.6), we expand $\mathbf{U}(\mathbf{z}-\mathbf{y})$ in a Taylor series about $\mathbf{x}$ and integrate term by term over the unit cell $\mathscr{V}_{\mathbf{x}}$ with the center $\mathbf{x}$, then

$$\langle\langle \mathbf{U} \rangle\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y}) = -\frac{1}{2\bar{\mathscr{V}}_{\mathbf{x}}}\int_{\mathscr{V}_{\mathbf{x}}} (\mathbf{z}-\mathbf{x}) \otimes (\mathbf{z}-\mathbf{x})\,\mathrm{d}\mathbf{z}\nabla\nabla\mathbf{U}(\mathbf{x}-\mathbf{y}) + \cdots. \quad (3.8)$$

As is evident from eqn (3.8), the tensor $\langle\langle \mathbf{U} \rangle\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y})$ is of order $O(a_{\mathscr{V}}^2|\mathbf{x}-\mathbf{y}|^{-5})$ with the dropped terms in eqn (3.8) of order $O(a_{\mathscr{V}}^4|\mathbf{x}-\mathbf{y}|^{-7})$ and higher-order terms. Then the absolute convergence of volume integral (3.6) takes the place because at sufficient distance $\mathbf{x}$ from the boundary ∂w and $|\mathbf{x}-\mathbf{y}| \to \infty$ the integration over $\mathbf{y}$ can be carried out independently for both $\langle\langle \mathbf{U} \rangle\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y})$ (the function of the 'slow' variable $\mathbf{x}-\mathbf{y}$) and the expression in curly brackets $\{\mathbf{L}_1(\mathbf{y})\mathbf{e}(\mathbf{y}) - [\mathbf{L}^{(0)} + \mathbf{L}_1(\mathbf{y})]\boldsymbol{\beta}_1(\mathbf{y})\}$ (the function of 'fast' variable $\mathbf{y}$) and therefore the volume integral converges absolutely. In a similar manner the term $\langle\langle \nabla\mathbf{G} \rangle\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{s})$ in the surface integral (3.6) is of order $O(a_{\mathscr{V}}^2|\mathbf{x}-\mathbf{y}|^{-4})$ and the surface integral vanishes at $|\mathbf{x}-\mathbf{s}| \to \infty$, $\mathbf{s} \in \partial w$. Expending $\mathbf{e}^0(\mathbf{z})$ in a Taylor series about $\mathbf{x}$ gives, by analogy with eqn (3.8), that $\langle\langle \mathbf{e}^0 \rangle\rangle_{\mathbf{x}}(\mathbf{x})$ is of order $O(a_{\mathscr{V}}^2\nabla\nabla\mathbf{e}^0(\mathbf{x}))$. Therefore, hereafter in interest of obtaining explicit final expressions, we can neglect by the term $\langle\langle \mathbf{e}^0 \rangle\rangle_{\mathbf{x}}(\mathbf{x})$ (3.6) as compared with $\langle \mathbf{e} \rangle_{\mathbf{x}}(\mathbf{x})$ in the 'slowly-varying' approximation of $\mathbf{e}^0(\mathbf{x})$.

By this means eqn (3.6) is reduced to the approximate relation (which is exact for the linear function $\mathbf{e}^0(\mathbf{x})$)

$$\mathbf{e}(\mathbf{x}) = \langle \mathbf{e} \rangle_{\mathbf{x}}(\mathbf{x}) + \int \langle\langle \mathbf{U} \rangle\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y})\{\mathbf{L}_1(\mathbf{y})\mathbf{e}(\mathbf{y}) - [\mathbf{L}^{(0)} + \mathbf{L}_1(\mathbf{y})]\boldsymbol{\beta}_1(\mathbf{y})\}\,\mathrm{d}\mathbf{y}. \quad (3.9)$$

Expressing eqn (3.9) in terms of stresses, the general equation

$$\boldsymbol{\sigma}(\mathbf{x}) = \langle \boldsymbol{\sigma} \rangle_{\mathbf{x}}(\mathbf{x}) + \int \langle\langle \boldsymbol{\Gamma} \rangle\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y})\boldsymbol{\eta}(\mathbf{y})\,\mathrm{d}\mathbf{y} \quad (3.10)$$

is obtained, where the 'strain polarization' tensor $\boldsymbol{\eta}$ defined as

$$\boldsymbol{\eta}(\mathbf{y}) = \mathbf{M}_1(\mathbf{y})\boldsymbol{\sigma}(\mathbf{y}) + \boldsymbol{\beta}_1(\mathbf{y}), \quad (3.11)$$

is simply a notational convenience. The integral operator kernel in eqn (3.10) is defined by the tensor $\mathbf{U}(\mathbf{x}-\mathbf{y})$ (3.5):

$$\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}) \equiv -\mathbf{L}^{(0)}[\mathbf{I}\delta(\mathbf{x}-\mathbf{y}) + \mathbf{U}(\mathbf{x}-\mathbf{y})\mathbf{L}^{(0)}], \quad (3.12)$$

where $\mathbf{I}$ is a unit fourth-order tensor.

Evidently the right-hand-side volume integrals in eqns (3.9) and (3.10) converge absolutely, during which no restrictions are imposed on the microtopology of the lattice Λ, and eqns (3.9) and (3.10) are valid for any deterministic (even nonperiodic) structures. The principle advantages of eqns (3.9) and (3.10) as compared with eqn (3.6) are the lack of the surface integral in eqns (3.9) and (3.10), and the local character of eqns (3.9) and (3.10). The last-mentioned advantage makes it possible to reduce the analysis of infinite number inclusion problems to the analysis of a finite number of inclusions located in some RVE (see Section 7 for details).

3.2. Some particular simplifications

In the interest of obtaining more simple relations, we now consider the case of triply periodic structures (2.9) under the uniform boundary conditions (2.10). Then the terms in curly brackets in the right-hand-side integrals of eqns (3.9) and (3.10) are invariant with respect to the unit cell number and eqns (3.9) and (3.10) can be rewritten in the form

$$\mathbf{e}(\mathbf{x}) = \mathbf{e}^{\Omega}(\mathbf{x}) + \int \mathbf{U}(\mathbf{x}-\mathbf{y})\{\mathbf{L}_1(\mathbf{y})\mathbf{e}(\mathbf{y}) - [\mathbf{L}^{(0)} + \mathbf{L}_1(\mathbf{y})]\boldsymbol{\beta}_1(\mathbf{y}) - (\mathbf{L}_1\mathbf{e} - \mathbf{L}\boldsymbol{\beta}_1)^{\Omega}(\mathbf{x})\}\,\mathrm{d}\mathbf{y}, \tag{3.13}$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}^{\Omega}(\mathbf{x}) + \int \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\{\boldsymbol{\eta}(\mathbf{y}) - \boldsymbol{\eta}^{\Omega}(\mathbf{x})\}\,\mathrm{d}\mathbf{y}. \tag{3.14}$$

In the pure elastic case $\boldsymbol{\beta} \equiv \mathbf{0}$, exact eqn (3.13) was previously used by Buryachenko and Parton (1992) for both $(\mathbf{L}_1\boldsymbol{\varepsilon})^{\Omega}(\mathbf{x}) \equiv$ const. and $\langle\boldsymbol{\varepsilon}\rangle_{\mathrm{x}}(\mathbf{x}) = \boldsymbol{\varepsilon}^{\Omega}(\mathbf{x}) \equiv$ const. when in eqn (3.14) $\langle\boldsymbol{\eta}\rangle_{\mathrm{x}}(\mathbf{x}) = \boldsymbol{\eta}^{\Omega}(\mathbf{x}) \equiv$ const. and $\langle\boldsymbol{\sigma}\rangle_{\mathrm{x}}(\mathbf{x}) = \boldsymbol{\sigma}^{\Omega}(\mathbf{x}) \equiv$ const. as well.

It should be emphasized that for the field X bounded in one direction [for example for the field X (2.6)] the surface integral (3.4) over a 'cylindrical' surface (with the surface area proportional to $\rho = |\mathbf{x}-\mathbf{s}|$) tends to zero with $|\mathbf{x}-\mathbf{s}| \to \infty$ as ρ^{-1} simply because the generalized function $\nabla\mathbf{G}(\mathbf{x}-\mathbf{s})$ is an even homogeneous function of order -2. Therefore, for infinite media the surface integral (3.4) vanishes, and eqn (3.4) can be rewritten as

$$\mathbf{e}(\mathbf{x}) = \mathbf{e}^{0}(\mathbf{x}) + \int \mathbf{U}(\mathbf{x}-\mathbf{y})\{\mathbf{L}_1(\mathbf{y})\mathbf{e}(\mathbf{y}) - [\mathbf{L}^{(0)} + \mathbf{L}_1(\mathbf{y})]\boldsymbol{\beta}_1(\mathbf{y})\}\,\mathrm{d}\mathbf{y}, \tag{3.15}$$

or, alternatively, in terms of stresses

$$\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}^{0}(\mathbf{x}) + \int \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\boldsymbol{\eta}(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{3.16}$$

Clearly in the considered case of X bounded in one direction, eqns (3.15) and (3.16) are exact, and the right-hand-side integrals in (3.15) and (3.16) converge absolutely.

Note that for the elastic analysis of triply periodic structures Fassi-Fehri et al. (1989) used eqn (3.15) (for $\boldsymbol{\beta} \equiv \mathbf{0}$) without any regularization of the right-hand-side integral in eqn (3.15) which diverges at infinity for the case of triply periodic composites.

3.3. Approximative effective field hypothesis

In order to simplify eqn (3.10) we now apply the main hypothesis of many micromechanical methods, the so-called effective field hypothesis (see for reference e.g. Buryachenko and Rammerstorfer, 1997):

(H1) Each inclusion v_i has an ellipsoidal form and is located in the field $\bar{\boldsymbol{\varepsilon}}_i \equiv \bar{\boldsymbol{\varepsilon}}(\mathbf{x})(\mathbf{x} \in v_i)$ which is homogeneous over the inclusion v_i. The perturbations introduced by the inclusion v_j at the point $\mathbf{x}$ are defined by the relations (no sum of j)

$$\int \mathbf{U}(\mathbf{x}-\mathbf{y}) V_j(\mathbf{y})\boldsymbol{\eta}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \mathbf{T}_j(\mathbf{x}-\mathbf{x}_j)\bar{v}_j\boldsymbol{\eta}_j, \tag{3.17}$$

where

$$\boldsymbol{\eta}_j \equiv \langle \boldsymbol{\eta}(\mathbf{x}) V_j(\mathbf{x})\rangle_{(j)} = \bar{v}_j^{-1} \int \boldsymbol{\eta}(\mathbf{x}) V_j(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{3.18}$$

is an average over the volume of the inclusion v_j, and

$$\mathbf{T}_j(\mathbf{x}-\mathbf{x}_j) = \begin{cases} \bar{v}_j^{-1} \int \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}) V_j(\mathbf{y})\,\mathrm{d}\mathbf{y} & \text{for } \mathbf{x}\notin v_j, \\ -\bar{v}_j^{-1}\mathbf{Q}_j & \text{for } \mathbf{x}\in v_j, \end{cases} \tag{3.19}$$

where the tensor $\mathbf{Q}_j$ is associated with the well-known Eshelby tensor by

$$\mathbf{S}_j = \mathbf{I}-\mathbf{M}^{(0)}\mathbf{Q}_j, \quad \mathbf{Q}_j \equiv -\langle \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\rangle_{(j)} = \text{const.}, \quad (\mathbf{x},\mathbf{y}\in v_j). \tag{3.20}$$

Then in the framework of the hypothesis (H1) and in view of the linearity of the problem there exist constant fourth and second-rank tensors $\mathbf{B}(\mathbf{x})$, $\mathbf{R}(\mathbf{x})$ and $\mathbf{C}(\mathbf{x})$, $\mathbf{F}(\mathbf{x})$, such that

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\bar{\boldsymbol{\sigma}}(\mathbf{x})+\mathbf{C}(\mathbf{x}), \quad \bar{v}_i\boldsymbol{\eta}(\mathbf{x}) = \mathbf{R}(\mathbf{x})\bar{\boldsymbol{\sigma}}(\mathbf{x})+\mathbf{F}(\mathbf{x}), \quad \mathbf{x}\in v_i, \tag{3.21}$$

where

$$\mathbf{R}(\mathbf{x}) = \bar{v}_i\mathbf{M}_1(\mathbf{x})\mathbf{B}(\mathbf{x}), \quad \mathbf{F}(\mathbf{x}) = \bar{v}_i[\mathbf{M}_1(\mathbf{x})\mathbf{C}(\mathbf{x})+\boldsymbol{\beta}(\mathbf{x})]. \tag{3.22}$$

According to Eshelby's (1961) theorem there are the relations between the averaged tensors (3.21)

$$\mathbf{R}(\mathbf{x}) = \bar{v}_i\mathbf{Q}_i^{-1}(\mathbf{I}-\mathbf{B}), \quad \mathbf{F} = -\bar{v}_i\mathbf{Q}_i^{-1}\mathbf{C}, \tag{3.23}$$

where $\mathbf{f}_i \equiv \langle \mathbf{f}(\mathbf{x})\rangle_{(i)}$ ($\mathbf{f}$ stands for $\mathbf{B}, \mathbf{C}, \mathbf{R}, \mathbf{F}$). No restrictions are imposed on the microtopology of the coated inclusions as well as on the inhomogeneity of the stress state in the coated inclusions. In the general case of coated inclusions v_i the tensors $\mathbf{B}(\mathbf{x})$ and $\mathbf{C}(\mathbf{x})$ can be found by the transformation method by Dvorak and Benveniste (1992) using finite element analysis. For particular cases of coated inclusions different analytical models are known (references can be found e.g. in the survey of Buryachenko and Rammerstorfer, 1998a, b).

For the homogeneous ellipsoidal domain v_i with

$$\mathbf{M}_1(\mathbf{x}) = \mathbf{M}_1^{(i)} = \text{const.}, \quad \boldsymbol{\beta}_1(\mathbf{x}) = \boldsymbol{\beta}_1^{(i)} = \text{const.} \quad \text{at } \mathbf{x}\in v_i, \tag{3.24}$$

we have a classical solution

$$\mathbf{B} = (\mathbf{I}+\mathbf{Q}_i\mathbf{M}_1^{(i)})^{-1}, \quad \mathbf{C} = -\mathbf{B}\mathbf{Q}_i\boldsymbol{\beta}_1^{(i)},$$

$$\mathbf{R} = \bar{v}_i\mathbf{M}_1^{(i)}\mathbf{B}, \quad \mathbf{F} = \bar{v}_i(\mathbf{I}+\mathbf{M}_1^{(i)}\mathbf{Q}_i)^{-1}\boldsymbol{\beta}_1^{(i)}. \tag{3.25}$$

4. Estimation of effective stress in the inclusions

4.1. Local approximation of effective stresses

In the framework of the hypothesis (H1) the system (3.10), taking (3.21) and (3.23) into account we get

$$\bar{\boldsymbol{\eta}}(\mathbf{x}_i) = \boldsymbol{\eta}^{\mathrm{av}}(\mathbf{x}_i) + \int \mathbf{K}(\mathbf{x}_i, \mathbf{y})\bar{\boldsymbol{\eta}}(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{4.1}$$

where $\bar{\boldsymbol{\eta}}(\mathbf{x}) \equiv \boldsymbol{\eta}_i$, and $\bar{v}_i\boldsymbol{\eta}^{\mathrm{av}}(\mathbf{x}) \equiv \mathbf{R}\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}}(\mathbf{x}) + \mathbf{F}$ (at $\mathbf{x} \in v_i$) will be named the modified strain polarization tensor of average stresses. The integral operator kernel introduced in eqn (4.1) is defined on the lattice Λ

$$\mathbf{K}(\mathbf{x}_i, \mathbf{y}) = \mathbf{R}\sum_{\mathbf{m}}[\mathbf{T}_{i\mathbf{m}}(\mathbf{x}_i - \mathbf{x}_{\mathbf{m}})(1 - V_i(\mathbf{y})) - \langle \mathbf{T}_{\mathbf{m}}\rangle_{\mathbf{x}_i}(\mathbf{x}_i - \mathbf{x}_{\mathbf{m}})]\delta(\mathbf{y} - \mathbf{x}_{\mathbf{m}}), \tag{4.2}$$

and

$$\mathbf{T}_{i\mathbf{m}}(\mathbf{x}_i - \mathbf{x}_{\mathbf{m}}) = (\bar{v}_i\bar{v}_{\mathbf{m}})^{-1}\int \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{y})V_i(\mathbf{x})V_{\mathbf{m}}(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}. \tag{4.3}$$

Rewriting eqn (4.1) in the spirit of a subtraction technique

$$\bar{\boldsymbol{\eta}}(\mathbf{x}_i) = \bar{\boldsymbol{\eta}}^{\mathrm{av}}(\mathbf{x}_i) + \int \mathbf{K}(\mathbf{x}_i, \mathbf{y})\,\mathrm{d}\mathbf{y}\bar{\boldsymbol{\eta}}(\mathbf{x}_i) + \int \mathbf{K}(\mathbf{x}_i), \mathbf{y})[\bar{\boldsymbol{\eta}}(\mathbf{y}) - \bar{\boldsymbol{\eta}}(\mathbf{x}_i)]\,\mathrm{d}\mathbf{y}, \tag{4.4}$$

gives

$$\bar{\boldsymbol{\eta}}(\mathbf{x}_i) = \mathbf{Y}(\mathbf{x}_i)\boldsymbol{\eta}^{\mathrm{av}}(\mathbf{x}_i) + \int \mathscr{K}(\mathbf{x}_i, \mathbf{y})\bar{\boldsymbol{\eta}}(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{4.5}$$

where

$$\mathbf{Y}(\mathbf{x}_i) \equiv (\mathbf{I} - \int \mathbf{K}(\mathbf{x}_i, \mathbf{y})\,\mathrm{d}\mathbf{y})^{-1}, \tag{4.6}$$

$$\mathscr{K}(\mathbf{x}_i, \mathbf{y}) \equiv \mathbf{Y}(\mathbf{x}_i)[\mathbf{K}(\mathbf{x}_i, \mathbf{y}) - \delta(\mathbf{x}_i - \mathbf{y})\int \mathbf{K}(\mathbf{x}_i, \mathbf{z})\,\mathrm{d}\mathbf{z}]. \tag{4.7}$$

The matrix $\mathbf{Y}(\mathbf{x}_i)$ determines the 'local' action of the surrounding inclusions on the separated one, while the integral operator kernel $\mathscr{K}(\mathbf{x}_i, \mathbf{y})$ describes a 'nonlocal' action of these inclusions. For the purpose of clarifying the above statement we consider, as an example, a particular problem by Buryachenko and Parton (1992) for triply periodic structures (2.9) under the homogeneous boundary conditions (2.10). In such a case they found that $\mathbf{Y}(\mathbf{x}_i) \equiv \mathbf{Y}^{\mathrm{tri}} = \mathrm{const.}$, and for an ellipsoidal representative volume element w^{el} (RVE) containing a statistically large number of inclusions the result can be rewritten in the current notation as

$$\mathbf{Y}^{\mathrm{tri}} = \left(\mathbf{I} - \mathbf{R}\mathbf{Q}(w^{\mathrm{el}}) - \mathbf{R}\sum_{\mathbf{m}\neq i}\mathbf{T}_{i\mathbf{m}}(\mathbf{x}_i - \mathbf{x}_{\mathbf{m}})\right)^{-1}, \quad \mathbf{x}_{\mathbf{m}} \in w^{\mathrm{el}}, \tag{4.8}$$

where the tensor $\mathbf{Q}(w^{\mathrm{el}})$ is defined for the domain w^{el} in a similar manner to eqn (3.20). Clearly for triply periodic structures (2.9) eqns (4.6) and (4.8) coincide. If furthermore, $\boldsymbol{\sigma}^0(\mathbf{x}) \equiv \mathrm{const.}$ then $\bar{\boldsymbol{\eta}}(\mathbf{y})$ is insensitive to translations, and the right-hand-side integral in eqn (4.5) vanishes, and eqn (4.5) is local.

When one of two (or both) assumptions (2.9) and (2.10) breaks down, eqn (4.5) is nonlocal on two counts. So, if only the assumption (2.10) breaks down then $\bar{\boldsymbol{\eta}}(\mathbf{y}) \neq \mathrm{const.}$, and the integral in eqn (4.5) does not vanish. Nevertheless the current kind of the nonlocalization can be usually easily analyzed. So, in this case the kernel $\mathscr{K}(\mathbf{x}_i, \mathbf{y})$ is a translation kernel: $\mathscr{K}(\mathbf{x}_i, \mathbf{y}) = \mathscr{K}(\mathbf{x}_i - \mathbf{y})$, and the current problem is akin to the estimation of nonlocal effects in statistically homogeneous random structure composites. So, for slowly-varying functions $\bar{\boldsymbol{\eta}}(\mathbf{y})$ Taylor expansion of $\bar{\boldsymbol{\eta}}(\mathbf{y})$ about $\mathbf{x}_i$ reduces eqn (4.5) to a differential equation with constant coefficients. The method of its solving that first comes to mind is using the Fourier transformation to transform the differential version of solving (4.5) into the inverse problem of solving the multiplicative equation (see e.g. Buryachenko, 1998, 1999; Buryachenko and Rammerstorfer, 1998c).

The breakdown of the assumption (2.9) is more common in practice, because it leads to the inequality $\mathbf{Y}(\mathbf{x}_i) \neq$ const. Then the average stresses $\langle \boldsymbol{\sigma} \rangle_{\mathbf{x}}(\mathbf{x}) \neq$ const. and hence $\bar{\boldsymbol{\eta}}(\mathbf{y}) \neq$ const. To put this another way, we will have a nonlocal eqn (4.5) even at the homogeneous boundary conditions (2.10), which is difficult to solve by the Fourier transform method insofar as $\mathscr{K}(\mathbf{x}_i, \mathbf{y}) \neq \mathscr{K}(\mathbf{x}_i - \mathbf{y})$. Moreover, let us consider $\mathbf{Y}(\mathbf{x}_i)$ (4.6) as an approximation of the corresponding right-hand-side nonlocal operator in eqn (4.5) by constant tensor (zero-order approximation). From such consideration it can be concluded that 'local' part $\mathbf{Y}(\mathbf{x}_i)$ of the nonlocal operator (4.5) depends explicitly not only on the local parameters of the inclusion distribution at the point $\mathbf{x}_i$, but also in a certain neighborhood of that point. This sort of a so-called nonlocal effect was identified by Buryachenko and Rammerstorfer (1998c) for statistically inhomogeneous random structure composites, and takes place for doubly-periodic structure as well (see also Section 7).

4.2. *Estimation of the nonlocal operator by the iteration method*

As mentioned the method of Fourier transform has been much investigated in nonlocal micromechanics, but its use is difficult if $\mathscr{K}(\mathbf{x}_i, \mathbf{y}) \neq \mathscr{K}(\mathbf{x}_i - \mathbf{y})$. This inconsistency can be avoided if the method of successive approximations, which is also called the Neumann series method, is used. With this in mind we initially define the function

$$(\mathscr{K}\bar{\eta})(\mathbf{x}_i) = \int \mathscr{K}(\mathbf{x}_i, \mathbf{y})\bar{\boldsymbol{\eta}}(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{4.9}$$

Then eqn (4.5) can be abbreviated as

$$\bar{\boldsymbol{\eta}} = \mathbf{Y}\boldsymbol{\eta}^{\mathrm{av}} + \mathscr{K}\bar{\eta}. \tag{4.10}$$

The iteration method proceeds by using the recursion formula

$$\bar{\boldsymbol{\eta}}_{(k+1)} = \mathbf{Y}\boldsymbol{\eta}^{\mathrm{av}} + \mathscr{K}\bar{\eta}_{(k)} \tag{4.11}$$

to construct a sequence of functions $\{\bar{\boldsymbol{\eta}}_{(k)}\}$ that can be treated as an approximation of the solution of eqn (4.10). Usually the driving term of this equation is used as an initial approximation:

$$\bar{\boldsymbol{\eta}}_{(0)}(\mathbf{x}_i) = \mathbf{Y}(\mathbf{x}_i)\boldsymbol{\eta}^{\mathrm{av}}(\mathbf{x}_i), \tag{4.12}$$

which is a local approximation of the effective stress in terms of the Subsection 4.1. The first approximation is

$$\bar{\boldsymbol{\eta}}_{(1)}(\mathbf{x}_i) = \mathbf{Y}(\mathbf{x}_i)\boldsymbol{\eta}^{\mathrm{av}}(\mathbf{x}_i) + \mathbf{Y}(\mathbf{x}_i)\int \mathbf{K}(\mathbf{x}_i, \mathbf{y})[\mathbf{Y}(\mathbf{y})\boldsymbol{\eta}^{\mathrm{av}}(\mathbf{y}) - \mathbf{Y}(\mathbf{x}_i)\boldsymbol{\eta}^{\mathrm{av}}(\mathbf{x}_i)]\,\mathrm{d}\mathbf{y}, \tag{4.13}$$

and again proceeding formally, it suggests the Neumann series form for the solution $\bar{\boldsymbol{\eta}}$ of (4.10)

$$\bar{\boldsymbol{\eta}} = \sum_{k=0}^{\infty} \mathscr{K}^k \mathbf{Y}\boldsymbol{\eta}^{\mathrm{av}}, \tag{4.14}$$

where the power $\mathscr{K}^k$ is defined recursively by the condition $\mathscr{K}^1 = \mathscr{K}$ and the kernel of $\mathscr{K}^k$ is (see e.g. Pipkin, 1991)

$$\mathscr{K}_k(\mathbf{x},\mathbf{y}) = \int \mathscr{K}(\mathbf{x},\mathbf{z})\mathscr{K}_{k-1}(\mathbf{z},\mathbf{y})\,\mathrm{d}\mathbf{z}. \tag{4.15}$$

In effect the iteration method (4.11) transforms the integral equation problem (4.5) into the linear algebra problem (4.14) and (4.15) in any case. The sequence $\{\bar{\boldsymbol{\eta}}_{(k)}\}$ (4.11) converges to a solution $\bar{\boldsymbol{\eta}}$ (4.5) for the kernel of $\mathscr{K}$ 'small' enough. The classical mathematical existence and uniqueness problems (and what is meant by saying 'small' or 'slowly-varying' enough), as are usually assumed in micromechanics, are beyond the scope of the current paper. Nevertheless, it should be mentioned that $\mathbf{Y}(\mathbf{x}_i) \to \mathbf{I}$ and $\int \mathscr{K}(\mathbf{x}_i,\mathbf{y})\,\mathrm{d}\mathbf{y} \to \mathbf{0}$ as $n(\mathbf{x}_j)\bar{v}_j \to 0\ \forall_j$. Therefore, the iteration method is appropriate at least for dilute concentration of inclusions. However, as we shall see in Section 7, it is easy to compute the approximation solution by the constructed procedure (4.14) used with the acceptable controlled accuracy.

5. Average stresses in the components and effective thermoelastic properties

5.1. General relations

Substituting (4.14) into (3.21) and combining terms, (3.21) finally gives the stress field inside the inclusions, $\boldsymbol{\sigma}(\mathbf{z})(\mathbf{z}\in v_i)$

$$\boldsymbol{\sigma}(\mathbf{x}_i,\mathbf{z}) = \mathbf{B}(\mathbf{z})\mathbf{R}^{-1}\left\{-\mathbf{F}+\bar{v}_i\sum_{n=0}^{\infty}(\mathscr{K}^n\mathbf{Y}\boldsymbol{\eta}^{\mathrm{av}})(\mathbf{x}_i)\right\}+\mathbf{C}(\mathbf{z}), \tag{5.1}$$

from which the representation for the average stress inside the inclusion v_i follows

$$\langle\boldsymbol{\sigma}\rangle_i = \mathbf{B}\mathbf{R}^{-1}\left\{-\mathbf{F}+\bar{v}_i\sum_{n=0}^{\infty}(\mathscr{K}^n\mathbf{Y}\boldsymbol{\eta}^{\mathrm{av}})(\mathbf{x}_i)\right\}+\mathbf{C}, \tag{5.2}$$

where the 'fast' variable $\mathbf{z}\in v_i$ characterizing the stress state is defined in the local coordinate system connected with the semiaxes of the ellipsoid v_i. There is connection between the 'slow' $\mathbf{x}$ and 'fast' $\mathbf{z}\in v_i$ variables: $\mathbf{x} = \Sigma m_j\mathbf{e}_j+\mathbf{z}$.

The mean stress in the matrix of the cell Ω_i follows simply from eqn (5.2) and the relation

$$\langle\boldsymbol{\sigma}\rangle_0(\mathbf{x}) = \frac{1}{c^{(0)}}(\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}}(\mathbf{x})-c^{(1)}\langle\boldsymbol{\sigma}\rangle_i), \tag{5.3}$$

where $\mathbf{x}\in\Omega_i\backslash v_i$. Substituting (5.2) into (3.10) gives the local stress in the matrix $\mathbf{x}\in\Omega_i\backslash v_i$

$$\boldsymbol{\sigma}(\mathbf{x}) = \langle\boldsymbol{\sigma}\rangle_{\mathbf{x}}(\mathbf{x})+\int\langle\langle\boldsymbol{\Gamma}\rangle\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y})\sum_{n=0}^{\infty}(\mathscr{K}^n\mathbf{Y}\boldsymbol{\eta}^{\mathrm{av}})(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{5.4}$$

Our goal is to find a constitutive equation relating $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x})$ to $\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}_i}(\mathbf{x})$, which is valid when these vary with $\mathbf{x}_i$. After estimating local stresses inside the inclusions, see (5.1), this problem becomes trivial, and, taking the average (2.8) or (2.3), leads to

$$\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i) = (\mathscr{M}^*\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}_i})(\mathbf{x}_i)+\mathscr{B}^*(\mathbf{x}_i), \tag{5.5}$$

where the integral operator $\mathscr{M}^*$ and the tensor $\mathscr{B}^*$ admit the representation

$$\mathscr{M}^* = \mathbf{M}^{(0)} + \mathscr{Y}\mathbf{R}, \tag{5.6}$$

$$\mathscr{B}^* = \boldsymbol{\beta}^0 + \mathscr{Y}\mathbf{F}, \tag{5.7}$$

and the operator $\mathscr{Y}$ is defined by the Neumann series

$$\mathscr{Y} = \sum_{k=0}^{\infty} n\mathscr{K}^k \mathbf{Y}. \tag{5.8}$$

The quantities $\mathscr{M}^*$ and $\mathscr{B}^*$ are called the effective compliance operator and the effective eigenstrains, respectively, and are simply a notational convenience.

It should be emphasized that both $\mathscr{M}^*$ (5.6) and $\mathscr{B}^*$ (5.7) are linear functions of the operator $\mathscr{Y}$ (5.8). Therefore, the effective eigenstrains $\mathscr{B}^*$ can be expressed in terms of the effective compliance operator $\mathscr{M}^*$:

$$\mathscr{B}^* = \boldsymbol{\beta}^{(0)} - \mathbf{M}^{(0)}\mathbf{R}^{-1}\mathbf{F} + \mathscr{M}^*\mathbf{R}^{-1}\mathbf{F}. \tag{5.9}$$

5.2. Some particular cases

For example, the zeroth-order approximation of eqn (5.5) is contained in the familiar constitutive equations for a homogeneous solid with material properties replaced by effective properties

$$\mathscr{M}^*_{(0)}(\mathbf{x}_i) = \mathbf{M}^*(\mathbf{x}_i) \equiv \mathbf{M}^{(0)} + \mathbf{Y}(\mathbf{x}_i)\mathbf{R}n(\mathbf{x}_i), \tag{5.10}$$

$$\mathscr{B}^*_{(0)}(\mathbf{x}_i) = \boldsymbol{\beta}^*(\mathbf{x}_i) \equiv \boldsymbol{\beta}^{(0)} + \mathbf{Y}(\mathbf{x}_i)\mathbf{F}n(\mathbf{x}_i). \tag{5.11}$$

In the first-order approximation of the constitutive eqn (5.5) additional integral terms involving average strains and stresses arise

$$\langle \boldsymbol{\varepsilon} \rangle_{\mathbf{x}_i}(\mathbf{x}_i) = (\mathscr{M}^*_{(1)}\langle \boldsymbol{\sigma} \rangle_{\mathbf{x}_i})(\mathbf{x}_i) + \mathscr{B}^*_{(1)}(\mathbf{x}_i), \tag{5.12}$$

where

$$(\mathscr{M}^*_{(1)}\langle \boldsymbol{\sigma} \rangle_{\mathbf{x}_i})(\mathbf{x}_i) = \mathbf{M}^*(\mathbf{x}_i)\langle \boldsymbol{\sigma} \rangle_{\mathbf{x}_i}(\mathbf{x}_i) + n(\mathbf{x}_i)\mathbf{Y}(\mathbf{x}_i)\int \mathbf{K}(\mathbf{x}_i, \mathbf{y})[\mathbf{Y}(\mathbf{y})\langle \boldsymbol{\sigma} \rangle_{\mathbf{y}}(\mathbf{y}) - \mathbf{Y}(\mathbf{x}_i)\langle \boldsymbol{\sigma} \rangle_{\mathbf{x}_i}(\mathbf{x}_i)]\, \mathrm{d}\mathbf{y}\mathbf{R}, \tag{5.13}$$

$$\mathscr{B}^*_{(1)}(\mathbf{x}_i) = \boldsymbol{\beta}^*(\mathbf{x}_i) + n(\mathbf{x}_i)\mathbf{Y}(\mathbf{x}_i)\int \mathbf{K}(\mathbf{x}_i, \mathbf{y})[\mathbf{Y}(\mathbf{y}) - \mathbf{Y}(\mathbf{x}_i)]\, \mathrm{d}\mathbf{y}\mathbf{F}. \tag{5.14}$$

Therefore, the average strains at a point are related to the average stresses at every point.

Substituting the zero-order approximation of both $\mathscr{M}^*$ (5.10) and $\mathscr{B}^*$ (5.11) into (5.9) leads to

$$\boldsymbol{\beta}^*(\mathbf{x}_i) = \boldsymbol{\beta}^{(0)} + (\mathbf{M}^*(\mathbf{x}_i) - \mathbf{M}^{(0)})\mathbf{R}^{-1}\mathbf{F}. \tag{5.15}$$

In particular, for triply periodic composites (2.9) (when $\boldsymbol{\beta}^*(\mathbf{x}) = \boldsymbol{\beta}^* = \text{const.}$ and $\mathbf{M}^*(\mathbf{x}) = \mathbf{M}^* = \text{const.}$) with homogeneous inclusions (3.24) the classical formula for two-phase statistically homogeneous composites by Levin (1967) and by Rosen and Hashin (1970) follows from the relation (5.15)

$$\boldsymbol{\beta}^* = \boldsymbol{\beta}^{(0)} + (\mathbf{M}^* - \mathbf{M}^{(0)})(\mathbf{M}^{(1)} - \mathbf{M}^{(0)})^{-1}(\boldsymbol{\beta}^{(1)} - \boldsymbol{\beta}^{(0)}). \tag{5.16}$$

However, in contrast to the case of statistically homogeneous composites (5.16), the formula (5.15) is not an exact one, because the initial relation (4.1) is obtained under some additional assumptions.

6. Doubly-periodic inclusion field in a finite stringer

For the general case of doubly periodic structures we obtained the representations of nonlocal operators (5.2) and (5.5) in terms of the Neumann series (5.8). Nevertheless, for a doubly-periodic array of inclusions (2.6) in a finite ply containing $2m^l+1$ layers of the inclusions ($f(m_3)\equiv 0$ at $|m_3|>m^l$) the problem can be solved immediately if the composite material is subjected to a stress gradient along the functionally graded direction $\mathbf{e}_3$. Then in the framework of the effective field hypothesis (3.15) the general exact eqn (3.16) is reduced to the linear system of $2m^l+1$ algebraic equations for $\bar{\boldsymbol{\eta}}(\mathbf{x_m})$ in terms of $\boldsymbol{\eta}^0(\mathbf{x_n})$

$$\bar{v}_{\mathbf{n}}\bar{\boldsymbol{\eta}}(\mathbf{x_n}) = \bar{v}_{\mathbf{n}}\boldsymbol{\eta}^0(\mathbf{x_n})+\mathbf{R}\sum_{\mathbf{m}\neq\mathbf{n}}\mathbf{T_{nm}}(\mathbf{x_n}-\mathbf{x_m})\bar{v}_{\mathbf{m}}\bar{\boldsymbol{\eta}}(\mathbf{x_m}), \tag{6.1}$$

where $\mathbf{x_m}=(x^1_{\mathbf{m}},x^2_{\mathbf{m}},x^3_{\mathbf{m}})$ and $\bar{\boldsymbol{\eta}}(\mathbf{x_m})\equiv\bar{\boldsymbol{\eta}}(x^3_{\mathbf{m}})$; $\bar{v}_{\mathbf{n}}\boldsymbol{\eta}^0(\mathbf{x_n})\equiv\bar{v}_{\mathbf{n}}\boldsymbol{\eta}^0(x^3_{\mathbf{n}})\equiv\mathbf{R}\langle\boldsymbol{\sigma}^0(\mathbf{x})\rangle_{\mathbf{n}}+\mathbf{F}$ is called the external strain polarization tensor.

Recognizing that in each inclusion layer $x_3=$ const. $\bar{\boldsymbol{\eta}}(\mathbf{x_n})\equiv\bar{\boldsymbol{\eta}}(\mathbf{x_m})$ for $\forall\mathbf{n},\mathbf{m}$ the system (6.1) has the finite number of unknowns, and we may express eqn (6.1) in the following compact form

$$\bar{v}_{\mathbf{n}}\boldsymbol{\eta}^0(\mathbf{x_n}) = \sum_{m_3=-m^l}^{m_3=m^l}\bar{v}_{\mathbf{m}}\mathbf{D}^{-1}_{n_3m_3}\bar{\boldsymbol{\eta}}(\mathbf{x_m}), \tag{6.2}$$

which tends to the solution

$$\bar{v}_{\mathbf{n}}\bar{\boldsymbol{\eta}}(\mathbf{x_n}) = \sum_{m_3=-m^l}^{m_3=m^l}\bar{v}_{\mathbf{m}}\mathbf{D}_{n_3m_3}\boldsymbol{\eta}^0(\mathbf{x_m}). \tag{6.3}$$

Here the inverse matrix $\mathbf{D}^{-1}$

$$(\mathbf{D}^{-1})_{n_3m_3} = \delta_{n_3m_3}\left\{\mathbf{I}-\mathbf{R}\sum_{m_1,m_2\neq 0}\mathbf{T_{nm}}(\mathbf{x_n}-\mathbf{x_m})\right\}-(1-\delta_{n_3m_3})\mathbf{R}\sum_{\mathbf{m},|m_3|\leqslant m^l}\mathbf{T_{nm}}(\mathbf{x_n}-\mathbf{x_m}), \tag{6.4}$$

where n_3, $m_3\doteq -m^l$, m^l. By virtue of the fact that $\bar{\boldsymbol{\eta}}(\mathbf{x_n})$ and $\boldsymbol{\eta}^0(\mathbf{x_n})$ do not vary in the layer $x^3_{n_3}=$ const. one bears in mind, for definiteness sake, that $n_1,n_2=0$ (6.3).

Substituting (3.21) into (6.3) gives the explicit relations for the stress in the inclusions $\mathbf{z}\in v_{\mathbf{n}}$:

$$\boldsymbol{\sigma}(\mathbf{x_n},\mathbf{z}) = \mathbf{B}(\mathbf{z})\mathbf{R}^{-1}\left\{-\mathbf{F}+\sum_{m_3=-m^l}^{m^l}\mathbf{D}_{n_3m_3}[\mathbf{R}\langle\boldsymbol{\sigma}^0\rangle_{(\mathbf{m})}+\mathbf{F}]\right\}+\mathbf{C}(\mathbf{z}), \tag{6.5}$$

and in the matrix $\mathbf{z}\in\Omega_{\mathbf{n}}\backslash v_{\mathbf{n}}$:

$$\boldsymbol{\sigma}(\mathbf{x_n},\mathbf{z}) = \boldsymbol{\sigma}^0(\mathbf{z})+\sum_{\mathbf{m}}\mathbf{T_m}(\mathbf{z}-\mathbf{x_m})\sum_{k_3=-m^l}^{m^l}\mathbf{D}_{m_3k_3}[\mathbf{R}\langle\boldsymbol{\sigma}^0\rangle_{(\mathbf{k})}(\mathbf{x_k})+\mathbf{F}]. \tag{6.6}$$

Obtained formulae (6.5) and (6.6) make possible detailed estimations of local stresses in the cell $\Omega_\mathbf{n}$. However, the explicit relations between the average values $\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}_n}(\mathbf{x}_\mathbf{n})$ and $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_n}(\mathbf{x}_\mathbf{n})$ is more convenient to use for the analysis of the overall response of the inclusion ply. With this in mind we will average eqns (2.3) and (3.16) over the cell $\Omega_\mathbf{n}$ taking eqns (2.8) and (3.19) into account

$$\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_\mathbf{n}}(\mathbf{x}_\mathbf{n}) = \mathbf{M}^{(0)}\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}_\mathbf{n}}(\mathbf{x}_\mathbf{n})+\boldsymbol{\beta}^{(0)}+n(\mathbf{x}_\mathbf{n})\bar{v}_\mathbf{n}\bar{\boldsymbol{\eta}}(\mathbf{x}_\mathbf{n}), \tag{6.7}$$

$$\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}_\mathbf{n}}(\mathbf{x}_\mathbf{n}) = \langle\boldsymbol{\sigma}^0\rangle_{\mathbf{x}_\mathbf{n}}(\mathbf{x}_\mathbf{n})+\sum_\mathbf{m}\langle\mathbf{T}_\mathbf{m}\rangle_{\mathbf{x}_\mathbf{n}}(\mathbf{x}_\mathbf{n}-\mathbf{x}_\mathbf{m})\bar{v}_\mathbf{m}\bar{\boldsymbol{\eta}}(\mathbf{x}_\mathbf{m}). \tag{6.8}$$

For simplicity we will approximate $\langle\boldsymbol{\sigma}^0\rangle_{\mathbf{x}_\mathbf{n}}(\mathbf{x}_\mathbf{n}) = \langle\boldsymbol{\sigma}^0(\mathbf{x})\rangle_{(\mathbf{n})}(\mathbf{x}\in v_\mathbf{n})$. Then, substitution of (6.1) into (6.8) gives

$$\mathbf{R}\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}_\mathbf{n}}(\mathbf{x}_\mathbf{n})+\mathbf{F} = \bar{v}_\mathbf{n}\bar{\boldsymbol{\eta}}(\mathbf{x}_\mathbf{n})-\mathbf{R}\sum_{\mathbf{m}\neq\mathbf{n}}\mathbf{T}_\mathbf{nm}(\mathbf{x}_\mathbf{n}-\mathbf{x}_\mathbf{m})\bar{v}_\mathbf{m}\bar{\boldsymbol{\eta}}(\mathbf{x}_\mathbf{m})+\mathbf{R}\sum_\mathbf{m}\langle\mathbf{T}_\mathbf{m}\rangle_{\mathbf{x}_\mathbf{n}}(\mathbf{x}_\mathbf{n}-\mathbf{x}_\mathbf{m})\bar{v}_\mathbf{m}\bar{\boldsymbol{\eta}}(\mathbf{x}_\mathbf{m}). \tag{6.9}$$

Solving the algebraic system (6.9) in terms of $\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}_\mathbf{n}}(\mathbf{x}_\mathbf{n})$ and substituting this solution into (6.7) give

$$\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_\mathbf{n}}(\mathbf{x}_\mathbf{n}) = \mathbf{M}^{(0)}\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}_\mathbf{n}}(\mathbf{x}_\mathbf{n})+\boldsymbol{\beta}^0+n(\mathbf{x}_\mathbf{n})\sum_{m_3=-m_l}^{m_l}\mathbf{Z}_{n_3m_3}[\mathbf{R}\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}_\mathbf{m}}(\mathbf{x}_\mathbf{m})+\mathbf{F}], \tag{6.10}$$

where the matrix $\mathbf{Z}^{-1}$ has the following elements $(\mathbf{Z}^{-1})_\mathbf{nm}$:

$$(\mathbf{Z}^{-1})_{n_3m_3} = (\mathbf{D}^{-1})_{n_3m_3}+\sum_{m_1,m_2}\mathbf{R}\langle\mathbf{T}_\mathbf{m}\rangle_{\mathbf{x}_\mathbf{n}}(\mathbf{x}_\mathbf{n}-\mathbf{x}_\mathbf{m}). \tag{6.11}$$

The nonlocal eqn (6.10) may be represented in a standard integral form

$$\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_\mathbf{n}}(\mathbf{x}_\mathbf{n}) = \int\mathscr{M}^*(\mathbf{x}_\mathbf{n},\mathbf{y})\langle\boldsymbol{\sigma}\rangle_\mathbf{y}(\mathbf{y})\,\mathrm{d}y^3+\mathscr{B}^*(\mathbf{x}_\mathbf{n}), \tag{6.12}$$

where the kernel $\mathscr{M}^*(\mathbf{x}_\mathbf{n},\mathbf{y})$ of the nonlocal operator and the effective eigenstrain $\mathscr{B}^*(\mathbf{x}_\mathbf{n})$, introduced in (6.12), are defined as

$$\mathscr{M}^*(\mathbf{x}_\mathbf{n},\mathbf{y}) = \mathbf{M}^{(0)}\delta(\mathbf{x}_\mathbf{n}-\mathbf{y})+n(\mathbf{x}_\mathbf{n})\sum_{m_3=-m_l}^{m_l}\delta(\mathbf{x}_\mathbf{m}-\mathbf{y})\mathbf{Z}_{n_3m_3}\mathbf{R}, \tag{6.13}$$

$$\mathscr{B}^*(\mathbf{x}_\mathbf{n}) = \boldsymbol{\beta}^0+n(\mathbf{x}_\mathbf{n})\sum_{m_3=-m_l}^{m_l}\mathbf{Y}_{n_3m_3}\mathbf{F}. \tag{6.14}$$

For triply periodic structures (2.9) and the homogeneous boundary conditions (2.10) the overall constitutive eqns (6.12)–(6.14) are reduced to eqn (5.5) with the effective material tensors (5.10) and (5.11). The same equivalence takes place also for central cells of the ply thick enough:

$$m^l \gg 1, \quad |m_3| \ll m^l. \tag{6.15}$$

Nevertheless, in the general case of the finite inclusion ply we have $\langle\boldsymbol{\sigma}\rangle_\mathbf{x}(\mathbf{x}) \neq \text{const.}$ even for homogeneous boundary conditions (2.10). So, from eqns (6.3) and (6.8) we find

$$\langle \boldsymbol{\sigma} \rangle_{\mathbf{x_n}}(\mathbf{x_n}) = \boldsymbol{\sigma}^0 + \sum_{\mathbf{m}} \langle \mathbf{T_m} \rangle_{\mathbf{x_n}}(\mathbf{x_n} - \mathbf{x_m}) \sum_{l_3 = -m^l}^{m^l} \mathbf{D}_{m_3 l_3}[\mathbf{R}\boldsymbol{\sigma}^0 + \mathbf{F}], \tag{6.16}$$

and therefore $\langle \boldsymbol{\sigma} \rangle_{\mathbf{x_n}}(\mathbf{x_n}) \neq$ const. even for $\boldsymbol{\sigma}^0 \equiv$ const.

It should be pointed out that the long-range effect takes place at the estimation of average stresses (6.16). This fact is explained by different behaviors at the infinity of summed functions $\mathbf{K}(\mathbf{x}_i, \mathbf{x}_j)$ (as $O(|\mathbf{x}_i - \mathbf{x}_j|^{-5})$) (4.2) and $\mathbf{T}_{ij}(\mathbf{x}_i - \mathbf{x}_j)$ (as $O(|\mathbf{x}_i - \mathbf{x}_j|^{-3})$) (4.3) in the representations of effective properties (5.13) and the stress concentrator tensor (6.16), respectively. Although it is well known that integral of the function $\mathbf{T}_{ij}(\mathbf{x}_i - \mathbf{x}_j)$ is not absolutely convergent in the whole Euclidean space R^3, we have no convergence problems for any finite thickness of the ply, since in the case considered the domain of integration is bounded in the direction $\mathbf{e}_3$.

By this means the initial problem with infinite numbers of the spherical inclusions (micro level) are reduced to the problem with $2m^l + 1$ layers (meso layer). At the microlevel, each layer is treated as a particulate composite of appropriate inclusion volume fraction, the matrix-inclusion microtopology of which is explicitly accounted for as well as the interactions of inclusions from different layers are considered. At the meso level the composite is viewed as consisting of alternating homogenized layers. As this takes place, even for identical inclusion layers the effective local and nonlocal parameters of homogenized layers change from layer to layer (see for comparison Buryachenko, 1998). These nontrivial dependences are explained by the interactions of inclusions from different layers and therefore the coupling of micro and meso levels is established explicitly.

7. Numerical results

7.1. Three-dimensional fields

Let us consider as an example a composite consisting of isotropic homogeneous components and having identical spherical inclusions $\mathbf{L}^{(i)} = (3k^{(i)}, 2\mu^{(i)}) \equiv 3k^{(i)}\mathbf{N}_1 + 2\mu^{(i)}\mathbf{N}_2$, $(\mathbf{N}_1 = \boldsymbol{\delta} \otimes \boldsymbol{\delta}/3$, $\mathbf{N}_2 = \mathbf{I} - \mathbf{N}_1)$. Let an inclusion ply (see Fig. 1) has a simple cubic (SC) lattice containing $2m^l + 1$ layers of inclusions. For central inclusion layers of the thick ply (6.15) the local effective properties $\mathbf{M}^*(\mathbf{x}_i) = \mathbf{M}^* \equiv$ const. (5.10) coincide with the properties of triply periodic structures and the tensor of effective moduli $\mathbf{L}^* \equiv (\mathbf{M}^*)^{-1}$ is characterized by three elastic moduli:

$$\begin{aligned} k_{13}^* &= \tfrac{1}{3} L_{3333}^* + \tfrac{2}{3} L_{1133}^*, \\ \mu_{13}^* &= L_{1313}^*, \\ \tilde{\mu}_{13}^* &= \tfrac{1}{2} L_{3333}^* - \tfrac{1}{2} L_{1133}^*, \end{aligned} \tag{7.1}$$

where the stiffness components are given with respect to a coordinate system whose base vectors are normal to the faces of the unit cell. In the interest of obtaining maximum difference between the effective properties, estimated by the different methods we will consider the examples for hard inclusions ($v^{(0)} = v^{(1)} = 0.3, \mu^{(1)}/\mu^{(0)} = 1000$) as well as for the voids ($\mathbf{L}^{(1)} \equiv \mathbf{0}$), and a number of values of the volume concentration of inclusions. For triply periodic SC arrays the local elastic moduli (7.1) are computed by analytical method by Sangani and Lu (1987), Nunan and Keller

Table 1
The overall elastic constants of SC arrays of voids in the thick ply: the components $(ij) = (13)$ and $(ij) = (12)$ for the boundary layer (4.6), (5.10); (SL) Sangani and Lu (1987), (H1) the proposed method for triply periodic structure (4.8), (5.10)

	Boundary layer						Central layer					
	$k^*_{ij}/k^{(0)}$		$\mu^*_{ij}/\mu^{(0)}$		$\tilde{\mu}^*_{ij}/\mu^{(0)}$		$k^*_{13}/k^{(0)}$		$\mu^*_{13}/\mu^{(0)}$		$\tilde{\mu}^*_{13}/\mu^{(0)}$	
c	13	12	13	12	13	12	SL	H1	SL	H1	SL	H1
0.10	0.77	0.77	0.81	0.81	0.84	0.84	0.77	0.77	0.82	0.81	0.84	0.84
0.20	0.60	0.60	0.65	0.64	0.71	0.72	0.60	0.60	0.67	0.64	0.72	0.72
0.30	0.47	0.47	0.50	0.50	0.59	0.61	0.46	0.47	0.55	0.47	0.61	0.61
0.40	0.36	0.37	0.39	0.39	0.49	0.51	0.36	0.36	0.47	0.38	0.50	0.51
0.50	0.27	0.28	0.29	0.30	0.38	0.41	0.24	0.28	0.38	0.29	0.39	0.41

Table 2
The overall elastic constants of SC arrays of rigid inclusions in the thick ply: the components $(ij) = (13)$ and $(ij) = (12)$ for the boundary layer (4.6), (5.10); (N) Nunan and Keller (1984) for $c = 0.1$–0.4, (K) Kushch (1987) for $c = 0.5$, (H1) the proposed method (4.8), (5.10)

	Boundary layer						Central layer					
	$k^*_{ij}/k^{(0)}$		$\mu^*_{ij}/\mu^{(0)}$		$\tilde{\mu}^*_{ij}/\mu^{(0)}$		$k^*_{13}/k^{(0)}$		$\mu^*_{13}/\mu^{(0)}$		$\tilde{\mu}^*_{13}/\mu^{(0)}$	
c	13	12	13	12	13	12	N/K	H1	N/K	H1	N/K	H1
0.10	1.18	1.18	1.22	1.21	1.26	1.27	1.18	1.18	1.22	1.21	1.27	1.27
0.20	1.40	1.40	1.46	1.46	1.64	1.68	1.41	1.40	1.46	1.45	1.70	1.69
0.30	1.68	1.69	1.76	1.75	2.17	2.29	1.71	1.69	1.77	1.74	2.35	2.32
0.40	2.05	2.07	2.15	2.15	2.91	3.19	2.17	2.07	2.25	2.12	3.74	3.21
0.50	2.54	2.61	2.69	2.73	3.77	4.27	3.50	2.61	3.14	2.67	6.49	4.33

(1984), and Kushch (1987) as well as by the formulae (4.8) and (5.10) (see Tables 1 and 2); here $v^{(i)} \equiv (3k^{(i)} - 2\mu^{(i)})/(6k^{(i)} - 2\mu^{(i)})$, $(i = 0, 1)$ is a Poisson ratio; in addition to the relations (7.1) the parameters k^*_{12}, μ^*_{12} and $\tilde{\mu}^*_{12}$ obtained by replacement of the index 3 in eqns (7.1) by the index 2 were estimated for the boundary layer $m_3 = m^l$, $m^l = 50$ (see Tables 1 and 2). For the boundary layer $m_3 = m^l$ the tensor of local effective moduli $\mathbf{L}^*(\mathbf{x})$ shows hexagonal symmetry.

The composite considered is essentially anisotropic, and the anisotropy of the elastic properties increases in an essential manner in the boundary layer. So, in Table 3 the value L^*_{1133}/L^0_{1133} and L^*_{3333}/L^0_{3333} are presented for a rigid inclusion ply with cubic-like structure and different numbers of inclusion layers: the single layer ($m^l = 0$), the boundary layer of triply layer ply ($x^3 = |\mathbf{e}_3|, m^l = 1$),

Table 3
The local overall elastic constants of SC arrays of rigid inclusions: (B_{00}) the single layer, (B_{13}) the boundary layer of triply layer ply, (B_{03}) the central layer of triply layer ply, (B) the central layer of the thick layer ply

	$L^*_{1133}/L^{(0)}_{1133}$				$L^*_{3333}/L^{(0)}_{3333}$			
c	B_{00}	B_{13}	B_{03}	B	B_{00}	B_{13}	B_{03}	B
0.10	1.145	1.142	1.139	1.139	1.212	1.209	1.213	1.213
0.20	1.307	1.292	1.275	1.275	1.472	1.491	1.513	1.512
0.30	1.496	1.458	1.412	1.412	1.815	1.866	1.927	1.930
0.40	1.738	1.662	1.568	1.570	2.267	2.375	2.512	2.506
0.50	2.097	1.986	1.850	1.844	2.824	3.007	3.248	3.267

Table 4
Normalized SIF k_I in a semi-infinite periodic collinear row of cracks: (H1) effective field hypothesis method (7.8); (I) the improved method (7.9); (R) exact solution by Rubinstein (1987), (K) the exact solution (7.10)

	$k_\mathrm{I}(-l)$			$k_\mathrm{I}(l)$			$k_\mathrm{I}(\infty \pm l)$		
$h(2l)-1$	H1	I	R	H1	I	R	H1	I	K
0.30	1.209	1.148	1.093	1.209	1.307	1.243	1.443	1.482	1.477
0.20	1.289	1.194	1.172	1.289	1.454	1.425	1.628	1.703	1.689
0.10	1.467	1.290	1.276	1.467	1.830	1.794	2.064	2.271	2.207

the central layer of triply layer ply ($x^3 = 0, m^l = 1$), and the central layer of the thick ply ($x^3 = 0, m^l = 30$).

As can be seen from Tables 1–3, in the interior of a thick layer ply ($m^l \to \infty$), sufficiently far away from its boundary, $\mathbf{L}^*(\mathbf{x})$ coincides with the effective moduli $\mathbf{L}^*$ for the triply periodic structure. Near the boundary of the ply the tensors of the effective moduli $\mathbf{L}^*(\mathbf{x})$ vary significantly within the boundary layer $x^3 = \pm m^l|\mathbf{e}_3|$ (boundary layer effect). There is a slight dependence of local overall effective moduli $\mathbf{L}^*(\mathbf{x})$ on the ply size m^l (scale effect).

We now come to the analysis of the nonlocal constitutive eqn (5.6) and (5.12). Let us for the sake of definiteness, the finite ply of rigid spherical inclusions is subjected to a known average stress $\langle\boldsymbol{\sigma}\rangle_\mathbf{x}$ along the functionally graded direction e_3 and only one component of $\langle\sigma_{ij}\rangle_\mathbf{x}$ ($i,j = 1,2,3$) differs from zero

$$\langle\boldsymbol{\sigma}\rangle_\mathbf{x}(\mathbf{x}) = f(x^3)\boldsymbol{\sigma}^{\mathrm{con}}, \tag{7.2}$$

$$\sigma^{\mathrm{con}}_{ij} = \text{const.} \neq 0, \quad \text{all other } \sigma^{\mathrm{con}}_{kl} = 0, \quad (kl \neq ij), \tag{7.3}$$

and

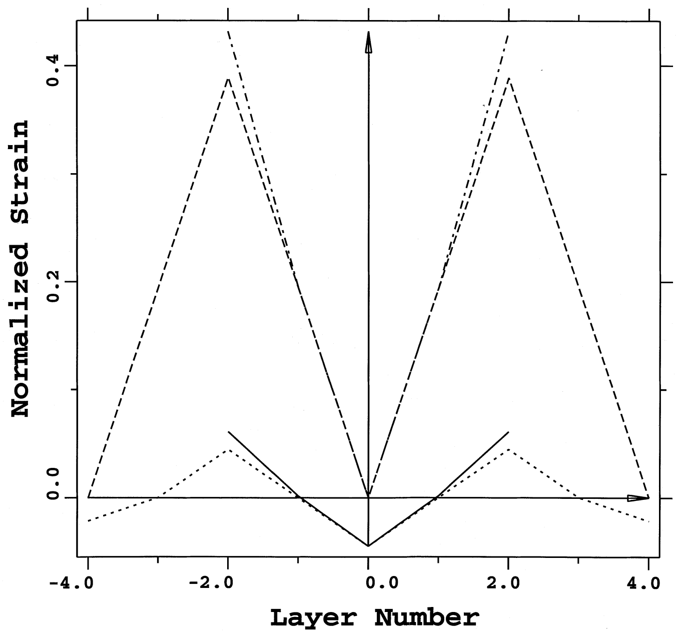

Fig. 2. Normalized local strains $\varepsilon_{33}^{\rm loc}(\mathbf{x})/(2q^{(0)}\sigma_{33}^{\rm con})$ (dashed and dot–dashed lines) and nonlocal $\varepsilon_{33}^{\rm non}(\mathbf{x})/(2q^{(0)}\sigma_{33}^{\rm con})$ (solid and dotted lines) components of strains vs a layer number to dimensionless parameter $a^{\rm nor}=2$: $m^l=2$ (solid and dot–dashed lines), $m^l=4$ (dotted and dashed lines).

$$f(x_3) = 1-\cos\left(\frac{\pi x^3}{a}\right), \tag{7.4}$$

where a is a positive length parameter. Let the tensors $\boldsymbol{\varepsilon}^{\rm loc}(\mathbf{x}) \equiv \mathbf{M}^*(\mathbf{x})\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}}(\mathbf{x})$ and $\boldsymbol{\varepsilon}^{\rm non}(\mathbf{x}) \equiv \langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x})-\mathbf{M}^*(\mathbf{x})\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}}(\mathbf{x})$ be named the local and nonlocal average strains, respectively. Clearly $f(2am)=0$ if m is the integer set Z, and therefore $\boldsymbol{\varepsilon}^{\rm loc}(\mathbf{x})=\mathbf{0}$ at $x^3=2am$, whereas $\boldsymbol{\varepsilon}^{\rm non}(\mathbf{x}) \neq \mathbf{0}$ at the mentioned points. In Fig. 2 the 33-components of normalized local $\varepsilon_{33}^{\rm loc}(x^3)/(2q^{(0)}\sigma_{33}^{\rm con})$ and nonlocal $\varepsilon_{33}^{\rm non}(x^3)/(2q^{(0)}\sigma_{33}^{\rm con})$ strains are plotted as the functions of integer-valued dimensionless coordinates $x^3/|\mathbf{e}_3| = 0, \pm 1, \pm 2, \ldots$ (a layer number) for different string thicknesses ($m^l=2$ and $m^l=4$) and for a single value of the normalised length parameter $a^{\rm nor} \equiv a/|\mathbf{e}_3| = 2$; here $\mathbf{M}^{(0)} \equiv (3p^{(0)}, 2q^{(0)})$, $c=0.5$. As can be seen from Fig. 2, the boundary layer effect shows up most vividly for nonlocal components of strains; so $\varepsilon_{33}^{\rm non}(\mathbf{x})$ in the boundary and central layers can differ from one another by a factor of two or even more, where the corresponding local components of strains differ only by 13%. The scale effect that takes place is small.

It should be mentioned that, in the computations reported in Fig. 2, the nonlocal average strains $\boldsymbol{\varepsilon}^{\rm non}(\mathbf{x})$ was evaluated using the first-order approximation of the nonlocal operator (5.13). Since we desire an evaluation of the accuracy of this approach, we will solve the same problem (5.5), (5.8) with any desirable accuracy by the truncation of the Neumann series. For the sake of definiteness we consider the central layers of the thick ply containing the rigid inclusions

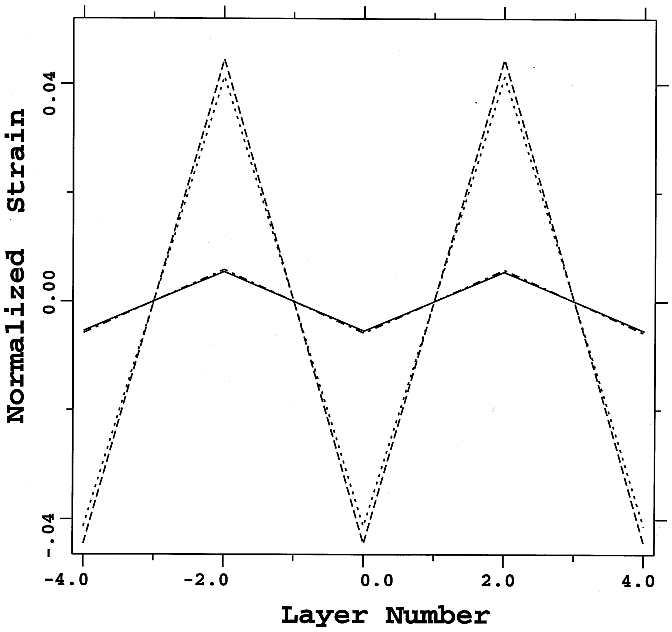

Fig. 3. Normalized components of nonlocal strains $\varepsilon_{ij}^{\text{non}}(\mathbf{x})/(2q^{(0)}\sigma_{ij}^{\text{con}})$ vs a layer number for both the first-order approximation (dot–dashed line is for $ij = 11$, dashed line is for $ij = 33$) and the twentieth-order approximation of the nonlocal operator $\mathcal{Y}$ (solid line is for $ij = 11$, dotted line is for $ij = 33$) to dimensionless parameter $a^{\text{nor}} = 2$.

($|m_3| \leqslant 4, m^l = 50, c = 0.5$) and loaded by the stress (7.2), (7.3). The results shown in Fig. 3 represent nonlocal parts of average strains $\boldsymbol{\varepsilon}^{\text{non}}(\mathbf{x})$ estimated by both the first (5.13) and twentieth ($k = 0, \ldots, 20$) (5.5), (5.6) and (5.8) order approximations of the nonlocal operator $\mathcal{Y}$ (5.8). The first-order approximation ($k = 0, 1$) (5.6) provides the accuracy within 7% error, the second-order approximation ($k = 0, 1, 2$) (5.6) guarantees the error no more than 0.5%. For the ply thick enough the maximum magnitude of local strains $\varepsilon_{33}^{\text{loc}}(\mathbf{x})$ is insensitive to the values of the normalized length parameter a^{nor} while the increase of a^{nor} leads to a decrease of the nonlocal component $\varepsilon_{33}^{\text{non}}(\mathbf{x})$ [compare Fig. 3 with Fig. 4, which is plotted for the twentieth iteration of the nonlocal operator (5.6) and (5.8)].

7.2. Two-dimensional fields

The method being proposed for the analysis of stress fields within doubly periodic structures (6.16) can be used for the consideration of some singly periodic structures if we assume that $|\mathbf{e}_2| \gg |\mathbf{e}_1|, |\mathbf{e}_3|$. Then the problem being analyzed is reduced to a problem for a single layer of inclusions periodical in the direction $|\mathbf{e}_1|$. The two-dimensional analog of this arrangement is a nonperiodic inclusion field located in one line. For the purpose of an evaluation of the accuracy of the proposed method we will consider an example which has an analytical solution obtained by Rubinstein (1987) by the method of the theory of functions of complex variables.

Namely, let us consider the plane problem of a semi-infinite regular grid of straight cuts (cracks) of the length $2l$ on line L ($x^2 = 0$) at nodes of a semi-infinite regular grid $x_n^1 = nh$

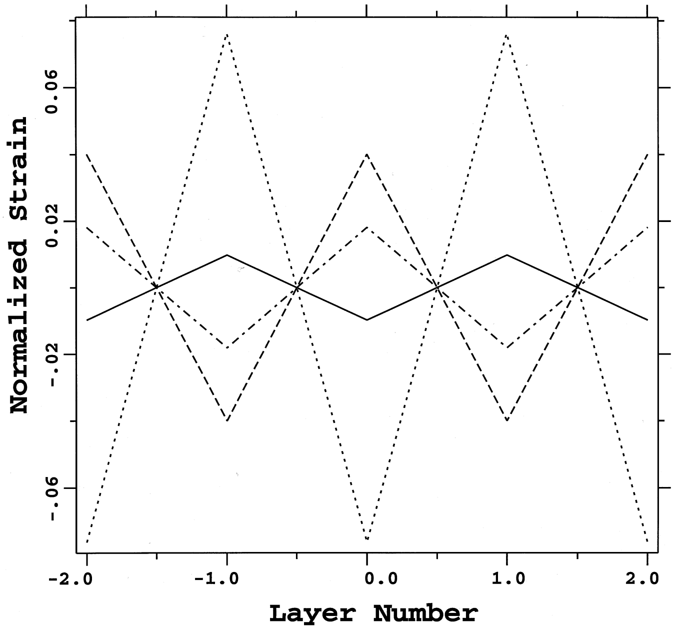

Fig. 4. Normalized components of nonlocal strains $\varepsilon_{ij}^{\rm non}(\mathbf{x})/(2q^{(0)}\sigma_{ij}^{\rm con})$ vs a layer number for the twentieth-order approximation of the nonlocal operator $\mathscr{Y}$ to dimensionless parameter $a^{\rm nor}=2$ (solid line is for $ij=11$, dotted line is for $ij=33$, dashed line is for $ij=12$, dot–dashed line is for $ij=13$).

($n\in Z^+\equiv\{0,1,\dots\}$; $h>2l$). In the case considered the method being analyzed in Section 6 is reduced to the method by Kachanov (1987), and we will use the analytical solution for two cracks presented in the paper mentioned above. The external field $\boldsymbol{\sigma}^0$ is uniaxial tension in the direction of the normal $\mathbf{n}\perp L$ and has the form $\sigma^0_{\alpha\beta}=\sigma^0_0 n_\alpha n_\beta$, σ^0_0 is a scalar, $\alpha,\beta=1,2$. Then the state of each defect is determined by the field $\bar{\boldsymbol{\sigma}}$ and $\bar{\sigma}_{\alpha\beta}n_\beta=\bar{\sigma}^0_0 n_\alpha$, where $\bar{\sigma}^0_0$ is a scalar. We estimate the relative change in the stress intensity factor (SIF) $k_{\rm I}=K_{\rm I}/K^0_{\rm I}$ vs h, where $K^0_{\rm I}=\sigma^0_0\sqrt{\pi l}$ is the SIF for an isolated crack in an unbounded plane. Then in the framework of the hypothesis (H1) we get

$$D^{\alpha\beta}_{nm}=D^0_{nm}n^\alpha n^\beta,\quad (D^0)^{-1}_{nm}\equiv\delta_{nm}-(1-\delta_{nm})T_{nm}, \tag{7.5}$$

$$T_{nm}(x^1_n-x^1_m)=\frac{\sqrt{|n-m|h}}{2l}\left(\sqrt{|n-m|h+2l}-\sqrt{|n-m|h-2l}\right), \tag{7.6}$$

$$T_m(x^1-x^1_m)=\frac{|(n-m)h-x^1|}{\sqrt{[(n-m)h-x^1]^2-l^2}}-1, \tag{7.7}$$

where $n\neq m$, and $x^1\notin[x^1_m-l,x^1_m+l]$.

In the simple case we adopt the estimation $k_{\rm I}=\bar{\sigma}^0_0/\sigma^0_0$, following from eqn (6.3)

$$k_{\rm I}(x^1_n\pm l)=\sum_{m=0}^{\infty}D^0_{nm}. \tag{7.8}$$

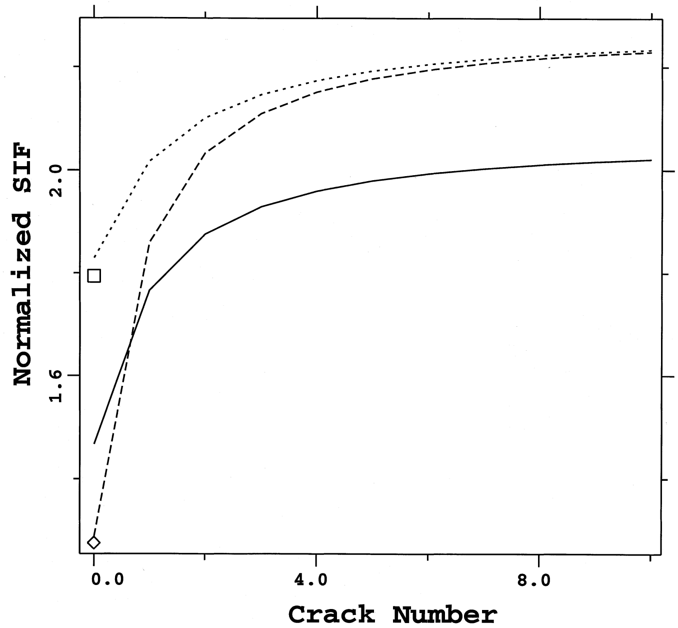

Fig. 5. Changes in normalized SIFs in a semi-infinite periodic collinear row of cracks vs a crack number. The solid line is for $k_{\rm I}$ (7.8), the dotted line is for $k_{\rm I}(x_n^3+l)$ (7.9), the dashed line is for $k_{\rm I}(x_n^3-l)$ (7.9). The symbols □ and ◇ denote the exact solution by Rubinstein (1987) for the right and left tips of the boundary crack ($x_n^1=0$), respectively. (SIFs are normalized to their values in absence of interactions. Spacing between cracks is 10% of the crack length.)

A more accurate expression for $k_{\rm I}$ can be constructed with consideration of the fact that the field $\bar{\sigma}_0^0 = \bar{\sigma}_0^0(x^1)(x^1 \in [x_n-l, x_n+l])$ is inhomogeneous in the neighborhood of the defect and equal to the superposition of the fields induced by the surrounding cracks:

$$k_{\rm I}(x_n^3 \pm l) = 1 + \frac{1}{\sqrt{\pi l}} \sum_{m \in Z_n^+} \int_{-l}^{l} \sqrt{\frac{l \pm \xi}{l \mp \xi}}\, T_m(\xi + x_n^1 - x_m^1)\, \mathrm{d}\xi \sum_{k=0}^{\infty} D_{mk}^0, \tag{7.9}$$

where Z_n^+ denotes the set $Z^+ \backslash n$. For the finite number of cracks as well as for the infinite regular grid of cracks ($x_n^1 = 0, \pm 1, \pm 2, \ldots$), eqn (7.9) is reduced to the relations analyzed previously by Kachanov (1987). Buryachenko and Parton (1990) obtained the relations similar to (7.9) by a more approximate method based on the consideration of triply interactive effects of cracks.

In Table 3 the exact solution by Koiter (1959) for the infinite regular grid of cracks ($x_n^1 = 0, \pm 1, \pm 2, \ldots$)

$$k_{\rm I}(\pm l) = \sqrt{\frac{h}{\pi l} \tan\left(\frac{\pi l}{h}\right)} \tag{7.10}$$

as well as the accurate solution for the boundary crack $k_{\rm I}(0 \pm l)$ obtained by Rubinstein (1987) are compared with the approximate solutions (7.8) and (7.9). The agreement of the approximate eqn (7.9) with the exact ones is satisfactory. Figure 5 shows that the boundary layer effect has a short range of influence: its impact is practically confined to the nearest four cracks. This particular example having an analytical solution was considered deliberately for the demonstration of the high accuracy of proposed method for the estimation of boundary layer effects in the degenerate case of FGMs. It should be mentioned that SIF depends essentially on the nonhomogeneity of the effective field $\bar{\boldsymbol{\sigma}}_0^0(\mathbf{x})$ leading to a significant difference of SIFs estimated by the formulae (7.8) and (7.9). In the estimation of both average stresses and effective properties of composites with ellipsoidal inclusions this dependence appears only slightly, and higher accuracy should be expected than using the effective field hypothesis (H1) (at least it was shown in the examples presented in Tables 1 and 2).

8. Conclusion

The solution obtained provides the calculation with reasonable accuracy for local and nonlocal elastic properties for a whole range of parameters. The method appears to be simple enough in both theoretical and computational aspects. Numerical calculations include only the use of lattice sums and the solution of well-conditioned linear algebraic systems.

Joint solution of the equilibrium equation, boundary conditions (2.5) and effective constitutive relations using either (6.10) or (6.12) leads to the estimation of average stresses $\langle\boldsymbol{\sigma}\rangle_x(\mathbf{x})$ and the average strains $\langle\boldsymbol{\varepsilon}\rangle_x(\mathbf{x})$. Of course, the mentioned scheme can be generalized easily to the case, where instead of each inclusion layer one considers an individual ply consisting of a few inclusion layers and different plies can be distinguished by the type of lattice periodicity as well as by the mechanical and geometrical parameters of the inclusions. In any case the effective properties of either the inclusion layers or the plies depend not only on the individual structure of the layer considered (as usual one assumes, see e.g. Plankensteiner et al., 1996) but on the parameters of other layers.

The obtained relations depend on the values associated with the mean distance between inclusions, and do not depend on the other characteristic size, i.e. the mean inclusion diameter. This fact may be explained by the initial use of the hypothesis H1 dealing with homogeneity of the field $\bar{\boldsymbol{\sigma}}(\mathbf{x})$ inside each inclusion. In the case of a variable representation of $\bar{\boldsymbol{\sigma}}(\mathbf{x})(\mathbf{x}\in v_i)$, for instance in polynomial form, the mean size of the inclusions will be contained in the nonlocal dependence of microstresses on the average stress $\langle\boldsymbol{\sigma}\rangle_x(\mathbf{x})$. Such an improvement was done in eqn (7.9) in comparison with eqn (7.8).

It should be mentioned that the effective constitutive eqn (5.5) was derived for points $\mathbf{x}_i$ located sufficiently far from the boundary of the body ∂w. In so doing the relations developed have been obtained by the use of the whole-space Green's function (3.3). Then use of nonlocal constitutive relations (5.5) requires more complicated boundary conditions (see Beran and McCoy, 1970; Drugan and Willis, 1996 for details); this question is beyond the scope of the current study.

The proposed method allows us to generalize the model to consider composites with any number of different components containing inclusions with different size, shape, orientation and properties, coated particles, cracks, etc. However, more detailed consideration of these facts are beyond the scope of the current paper.

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